SIXTH TERM EXAMINATION PAPER

administered by the Oxford and Cambridge Schools Examination Board on behalf of the Cambridge Colleges

9470

FURTHER MATHEMATICS

PAPER A

Thursday 27 June 1991, afternoon

3 hours

All questions carry equal weight.

Candidates are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

Candidates may attempt as many questions as they wish but marks will be assessed on the **six** *questions best answered.*

Mathematical Formulae and tables MF(STEP 1) are provided.

Electronic calculators may be used.

This Question Paper consists of 8 printed pages.

1 Let $h(x) = ax^2 + bx + c$, where a, b and c are constants, and $a \neq 0$. Give a condition which a, b and c must satisfy in order that $h(x)$ can be written in the form

$$a(x+k)^2, \qquad (*)$$

where k is a constant.

If $\mathrm{f}(x) = 3x^2 + 4x$ and $\mathrm{g}(x) = x^2 - 2$, find the two constant values of λ such that $\mathrm{f}(x) + \lambda\mathrm{g}(x)$ can be written in the form $(*)$. Hence, or otherwise, find constants A, B, C, D, m and n such that

$$\begin{aligned} \mathrm{f}(x) &= A(x+m)^2 + B(x+n)^2 \\ \mathrm{g}(x) &= C(x+m)^2 + D(x+n)^2. \end{aligned}$$

If $\mathrm{f}(x) = 3x^2 + 4x$ and $\mathrm{g}(x) = x^2 + \alpha$ and it is given that there is only one value of λ for which $\mathrm{f}(x) + \lambda\mathrm{g}(x)$ can be written in the form $(*)$, find α.

2 The equation of a hyperbola (with respect to axes which are displaced and rotated with respect to the standard axes) is

$$3y^2 - 10xy + 3x^2 + 16y - 16x + 15 = 0. \qquad (\dagger)$$

By differentiating $(\dagger)$, or otherwise, show that the equation of the tangent through the point (s,t) on the curve is

$$y = \left(\frac{5t - 3s + 8}{3t - 5s + 8}\right) x - \left(\frac{8t - 8s + 15}{3t - 5s + 8}\right).$$

Show that the equations of the asymptotes (the limiting tangents as $s \to \infty$) are

$$y = 3x - 4 \qquad \text{and} \qquad 3y = x - 4.$$

[**Hint:** you will need to find a relationship between s and t which is valid in the limit as $s \to \infty$.]

Show that the angle between one asymptote and the x-axis is the same as the angle between the other asymptote and the y-axis. Deduce the slopes of the lines that bisect the angles between the asymptotes and find the equations of the axes of the hyperbola.

3 It is given that x, y and z are distinct and non-zero, and that they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Show that $x^2y^2z^2 = 1$ and that the value of $x + \dfrac{1}{y}$ is either $+1$ or -1.

4 Let $y = \cos\phi + \cos 2\phi$, where $\phi = \dfrac{2\pi}{5}$. Verify by direct substitution that y satisfies the quadratic equation $2y^2 = 3y + 2$ and deduce that the value of y is $-\frac{1}{2}$.

Let $\theta = \dfrac{2\pi}{17}$. Show that

$$\sum_{k=0}^{16} \cos k\theta = 0.$$

If $z = \cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta$, show that the value of z is $-(1 - \sqrt{17})/4$.

5 Give a rough sketch of the function $\tan^k\theta$ for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$ in the two cases $k = 1$ and $k \gg 1$ (i.e. k is much greater than 1).

Show that for any positive integer n

$$\int_0^{\pi/4} \tan^{2n+1}\theta \, \mathrm{d}\theta = (-1)^n \left(\tfrac{1}{2}\ln 2 + \sum_{m=1}^{n} \frac{(-1)^m}{2m} \right),$$

and deduce that

$$\sum_1^{\infty} \frac{(-1)^{m-1}}{2m} = \tfrac{1}{2}\ln 2.$$

Show similarly that

$$\sum_1^{\infty} \frac{(-1)^{m-1}}{2m-1} = \frac{\pi}{4}.$$

6 Show by means of a sketch, or otherwise, that if $0 \leqslant \mathrm{f}(y) \leqslant \mathrm{g}(y)$ for $0 \leqslant y \leqslant x$ then

$$0 \leqslant \int_0^x \mathrm{f}(y)\mathrm{d}y \leqslant \int_0^x \mathrm{g}(y)\mathrm{d}y.$$

Starting from the inequality $0 \leqslant \cos y \leqslant 1$, or otherwise, prove that if $0 \leqslant x \leqslant \frac{1}{2}\pi$ then $0 \leqslant \sin x \leqslant x$ and $\cos x \geqslant 1 - \frac{1}{2}x^2$. Deduce that

$$\frac{1}{1800} \leqslant \int_0^{\frac{1}{10}} \frac{x}{(2 + \cos x)^2}\,\mathrm{d}x \leqslant \frac{1}{1797}.$$

Show further that if $0 \leqslant x \leqslant \frac{1}{2}\pi$ then $\sin x \geqslant x - \frac{1}{6}x^3$. Hence prove that

$$\frac{1}{3000} \leqslant \int_0^{\frac{1}{10}} \frac{x^2}{(1 - x + \sin x)^2}\,\mathrm{d}x \leqslant \frac{2}{5999}.$$

7 The function g satisfies, for all positive x and y,

$$\mathrm{g}(x) + \mathrm{g}(y) = \mathrm{g}(z), \qquad (*)$$

where $z = xy/(x + y + 1)$. By treating y as a constant, show that

$$\mathrm{g}'(x) = \frac{y^2 + y}{(x + y + 1)^2}\mathrm{g}'(z) = \frac{z(z+1)}{x(x+1)}\mathrm{g}'(z),$$

and deduce that $2\mathrm{g}'(1) = (u^2 + u)\mathrm{g}'(u)$ for all u satisfying $0 < u < 1$. Now by treating u as a variable, show that

$$\mathrm{g}(u) = A \ln\left(\frac{u}{u+1}\right) + B,$$

where A and B are constants. Verify that g satisfies $(*)$ for a suitable value of B. Can A be determined from $(*)$?

The function f satisfies, for all positive x and y,

$$\mathrm{f}(x) + \mathrm{f}(y) = \mathrm{f}(z)$$

where $z = xy$. Show that $\mathrm{f}(x) = C \ln x$ where C is a constant.

8 Solve the quadratic equation $u^2 + 2u \sinh x - 1 = 0$, giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

which satisfies $y = 0$ and $y' > 0$ at $x = 0$.

Find the solution of the differential equation

$$\sinh x \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x = 0$$

which satisfies $y = 0$ at $x = 0$.

9 Let G be the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b and c are integers modulo 5, and $a \neq 0 \neq c$. Show that G forms a group under matrix multiplication (which may be assumed to be associative). What is the order of G? Determine whether or not G is commutative.

Determine whether or not the set consisting of all elements in G of order 1 or 2 is a subgroup of G.

10 A straight stick of length h stands vertically. On a sunny day, the stick casts a shadow on flat horizontal ground. In cartesian axes based on the centre of the Earth, the position of the Sun may be taken to be $R(\cos\theta, \sin\theta, 0)$ where θ varies but R is constant. The positions of the base and tip of the stick are $a(0, \cos\phi, \sin\phi)$ and $b(0, \cos\phi, \sin\phi)$, respectively, where $b - a = h$. Show that the displacement vector from the base of the stick to the tip of the shadow is

$$Rh(R\cos\phi\sin\theta - b)^{-1}\begin{pmatrix} -\cos\theta \\ -\sin^2\phi\sin\theta \\ \cos\phi\sin\phi\sin\theta \end{pmatrix}.$$

['Stands vertically' means that the centre of the Earth, the base of the stick and the tip of the stick are collinear; 'horizontal' means perpendicular to the stick.]

11 The Ruritanian army is supplied with shells which may explode at any time in flight but not before the shell reaches its maximum height. The effect of the explosion on any observer depends only on the distance between the exploding shell and the observer (and decreases with distance). Ruritanian guns fire the shells with fixed muzzle speed, and it is the policy of the gunners to fire the shells at an angle of elevation which minimises the possible damage to themselves (assuming the ground is level) – i.e. they aim so that the point on the descending trajectory that is nearest to them is as far away as possible. With that intention, they choose the angle of elevation that minimises the damage to themselves if the shell explodes at its maximum height. What angle do they choose?

Does the shell then get any nearer to the gunners during its descent?

12 A particle is attached to one end B of a light elastic string of unstretched length a. Initially the other end A is at rest and the particle hangs at rest at a distance $a + c$ vertically below A. At time $t = 0$, the end A is forced to oscillate vertically, its downwards displacement at time t being $b \sin pt$. Let $x(t)$ be the downwards displacement of the particle at time t from its initial equilibrium position. Show that, while the string remains taut, $x(t)$ satisfies

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -n^2(x - b \sin pt),$$

where $n^2 = g/c$, and that if $0 < p < n$, $x(t)$ is given by

$$x(t) = \frac{bn}{n^2 - p^2}(n \sin pt - p \sin nt).$$

Write down a necessary and sufficient condition that the string remains taut throughout the subsequent motion, and show that it is satisfied if $pb < (n - p)c$.

13 A non-uniform rod AB of mass m is pivoted at one end A so that it can swing freely in a vertical plane. Its centre of mass is a distance d from A and its moment of inertia about any axis perpendicular to the rod through A is mk^2. A small ring of mass αm is free to slide along the rod and the coefficient of friction between the ring and rod is μ. The rod is initially held in a horizontal position with the ring a distance x from A. If $k^2 > xd$, show that when the rod is released, the ring will start to slide when the rod makes an angle θ with the downward vertical, where

$$\mu \tan\theta = \frac{3\alpha x^2 + k^2 + 2xd}{k^2 - xd}.$$

Explain what will happen if (i) $k^2 = xd$ and (ii) $k^2 < xd$.

14 The current in a straight river of constant width h flows at uniform speed αv parallel to the river banks, where $0 < \alpha < 1$. A boat has to cross from a point A on one bank to a point B on the other bank directly opposite to A. The boat moves at constant speed v relative to the water. When the position of the boat is (x, y), where x is the perpendicular distance from the opposite bank and y is the distance downstream from AB, the boat is pointing in a direction which makes an angle θ with AB. Determine the velocity vector of the boat in terms of v, θ and α.

The pilot of the boat steers in such a way that the boat always points exactly towards B. Show that the velocity vector of the boat is

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \tan\theta \dfrac{\mathrm{d}x}{\mathrm{d}t} + x\sec^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}t} \end{pmatrix}.$$

By comparing this with your previous expression deduce that

$$\alpha \frac{\mathrm{d}x}{\mathrm{d}\theta} = -x\sec\theta$$

and hence show that

$$(x/h)^\alpha = (\sec\theta + \tan\theta)^{-1}.$$

Let $s(t)$ be a new variable defined by $\tan\theta = \sinh(\alpha s)$. Show that $x = he^{-s}$, and that

$$he^{-s}\cosh(\alpha s)\frac{\mathrm{d}s}{\mathrm{d}t} = v.$$

Hence show that the time of crossing is $hv^{-1}(1-\alpha^2)^{-1}$.

15 Integers $n_1, n_2, \ldots, n_r$ (possibly the same) are chosen independently at random from the integers $1, 2, 3, \ldots, m$. Show that the probability that $|n_1 - n_2| = k$, where $1 \leqslant k \leqslant m-1$, is $2(m-k)/m^2$ and show that the expectation of $|n_1 - n_2|$ is $(m^2-1)/(3m)$. Verify, for the case $m = 2$, the result that the expectation of $|n_1 - n_2| + |n_2 - n_3|$ is $2(m^2-1)/(3m)$. Write down the expectation, for general m, of

$$|n_1 - n_2| + |n_2 - n_3| + \cdots + |n_{r-1} - n_r|.$$

Desks in an examination hall are placed a distance d apart in straight lines. Each invigilator looks after one line of m desks. When called by a candidate, the invigilator walks to that candidate's desk, and stays there until called again. He or she is equally likely to be called by any of the m candidates in the line but candidates never call simultaneously or while the invigilator is attending to another call. At the beginning of the examination the invigilator stands by the first desk. Show that the expected distance walked by the invigilator in dealing with $N+1$ calls is

$$\frac{d(m-1)}{6m}[2N(m+1)+3m].$$

16 Each time it rains over the Cabbibo dam, a volume V of water is deposited, almost instantaneously, in the reservoir. Each day (midnight to midnight) water flows from the reservoir at a constant rate u units of volume per day. An engineer, if present, may choose to alter the value of u at any midnight.

(i) Suppose that it rains at most once in any day, that there is a probability p that it will rain on any given day and that, if it does, the rain is equally likely to fall at any time in the 24 hours (i.e. the time at which the rain falls is a random variable uniform on the interval [0,24]). The engineer decides to take two days' holiday starting at midnight. If at this time the volume of water in the reservoir is V below the top of the dam, find an expression for u such that the probability of overflow in the two days is Q, where $Q < p^2$.

For the engineer's summer holidays, which last 18 days, the reservoir is drained to a volume kV below the top of the dam and the rate of outflow u is set to zero. The engineer wants to drain off as little as possible, consistent with the requirement that the probability that the dam will overflow is less than $\frac{1}{10}$. In the case $p = \frac{1}{3}$, find by means of a suitable approximation the required value of k.

(ii) Suppose instead that it may rain at most once before noon and at most once after noon each day, that the probability of rain in any given half-day is $\frac{1}{6}$ and that it is equally likely to rain at any time in each half day. Is the required value of k lower or higher?